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Building Bisimple Idempotent-Generated Semigroups

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In "The Fundamental Four-Spiral Semigroup" (*J. Algebra* **54** (1978), 6–26), the authors introduced and analyzed the structure of the fundamental four-spiral semigroup Sp_4 as an example of a bisimple idempotent-generated semigroup which is not completely simple. To measure the extent to which Sp_4 is a basic building block for the class $\mathscr{B}$ of bisimple non-completely simple idempotent-generated semigroups the following embedding question was raised: does every member of $\mathscr{B}$ contain an isomorphic copy of Sp_4 as a subsemigroup? In this paper we introduce several methods for building bisimple idempotent-generated semigroups and answer the embedding question in the negative. We provide several general techniques for constructing biordered sets and illustrate many of the resulting biordered sets diagramatically.

1. PRELIMINARY RESULTS, NOTATION

We shall assume familiarity with the standard notation and terminology of semigroup theory as presented in the book of Clifford and Preston [2]: in addition we shall assume that the reader is familiar with the notion of a (regular) *biordered set* $(E, \omega^r, \omega^l, \tau)$ as introduced by Nambooripad [7, 8] and in particular with his definition of the *sandwich set* $S(e, f)$ of $e, f \in E$. All biordered sets which we shall consider will be regular, so we shall drop the adjective "regular" in this paper. We denote the (biordered) set of idempotents of the regular semigroup S by $E(S)$. If E is a biordered set then a sequence $(e_0, e_1, ..., e_n)$ of elements $e_i \in E$ with $e_i(\mathscr{R} \cup \mathscr{L})\, e_{i+1}$ is called an E-sequence. The E-sequence $(e_0, e_1, ..., e_n)$ with $e_i \neq e_{i+1}$ is called an *E-chain* if $(e_i, e_{i+1}) \in \mathscr{R}[\mathscr{L}]$ implies $(e_{i+1}, e_{i+2}) \in \mathscr{L}[\mathscr{R}]$ for $i = 0, ..., n-2$. The *length* of the *E-chain* $(e_0, e_1, ..., e_n)$ is n (the number of

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ω-principal semigroups and in section 5 we show how to build *BIG* ω-principal semigroups from two copies of Sp_4. From Proposition 1.1 it follows that in order to find a counter-example to the embedding question we need to build a connected biordered set E for which the natural order relation ω is not trivial and in which the minimum distance between distinct comparable idempotents is greater than four. From Corollary 4.7 of [1], it follows that such a biordered set cannot be ω-principal. We exhibit two classes of counter-examples to the embedding question in section 3: These counter-examples are constructed as special cases of two general constructions of biordered sets which we introduce in section 2.

We note that the principal ideals of ω-principal biordered sets are semilattices (in fact chains). Such biordered sets arise from ordered structures called pseudo-semilattices: recall first that if X is a set and ρ is a relation on X, then for each $x \in X$, $\rho(x) = \{y \in X \mid y\rho x\}$.

DEFINITION 1.3 (Schein [17]). A *pseudo-semilattice* is a structure (E, ω^l, ω^r) consisting of a set E and two quasi-orders ω^l and ω^r such that to every pair e, $f \in E$ there is a unique element $e \wedge f \in E$ for which $\omega^l(e) \cap \omega^r(f) = w(e \wedge f)$. (Here $\omega = \omega^l \cap \omega^r$.)

Nambooripad [12] has introduced the concept of a partially associative pseudo-semilattice to elucidate the connection between pseudo-semilattices and biordered sets.

DEFINITION 1.4 (Nambooripad [12]). A pseudo-semilattice E is called *partially associative* if the following axioms and their duals hold (for $e, f, g \in E$):

(PA1) if $e\omega^r f$ then $e \wedge f = e$:

(PA2) let $f, g \in \omega^r(e)$: then

(i) $f \wedge (e \wedge g) = f \wedge g$, and

(ii) $e \wedge (f \wedge g) = (e \wedge f) \wedge g$.

THEOREM 1.5 (Nambooripad [12]). *Let $E = (E, \omega^l, \omega^r)$ be a pseudo-semilattice. If E is partially associative then there exists a unique family τ of partial transformations of E given by*

$$f\tau^r(e) = e \wedge f \qquad \forall f \in \omega^r(e)$$
$$f\tau^l(e) = f \wedge e \qquad \forall f \in \omega^l(e)$$

such that $(E, \omega^r, \omega^l, \tau)$ is a biordered set. Conversely if τ exists such that $(E, \omega^r, \omega^l, \tau)$ is a biordered set, then the pseudo-semilattice (E, ω^l, ω^r) is partially associative.

A regular semigroup S is called *pseudo-inverse* if $E(S)$ is a partially associative pseudo-semilattice. There are several equivalent ways of describing this, some of which are collected in the following theorem due to Nambooripad. Recall links in the chain): if $e, f \in E$ then $d(e, f)$ (the distance between e and f) is the length of a shortest E-chain ($e = e_0, e_1, \ldots, e_n = f$) linking e and f if such a chain exists: otherwise $d(e, f)$ is infinite. The biordered set E is called *connected* if $d(e, f)$ is finite for all $e, f \in E$. From [1] we have the following results.

PROPOSITION 1.1 [1]. *Let E be a biordered set. Then E is connected if and only if E is the biordered set of idempotents of some bisimple idempotent-generated semigroup. If $e, f \in E$, $e \neq f$ and $e\omega f$, then $d(e, f) \geqslant 4$.*

We shall refer to idempotent-generated semigroups as *IG* semigroups and to bisimple idempotent-generated semigroups as *BIG* semigroups in this paper.

The *fundamental four-spiral semigroup Sp_4* was introduced in [1] as the quotient of the free semigroup $\mathscr{F}_X$, where $X = \{a, b, c, d\}$, by the congruence ρ generated by the relation $\rho_0 = \{(a, a^2), (b, b^2), (c, c^2), (d, d^2), (a, ba), (b, ab), (b, bc), (c, cb), (c, dc), (d, cd), (d, da)\}$. The structure of Sp_4 was described as follows:

THEOREM 1.2 [1]. *Sp_4 is a bisimple fundamental (in fact combinatorial) regular idempotent-generated semigroup which is isomorphic to a rectangular band of four semigroups A, B, C and $D \cup E$ where A, B, C, D are bicyclic semigronps and E is infinite cyclic.*

It was further shown that the biordered set of Sp_4 is isomorphic to the *four-spiral biordered set E_4* which is pictured in diagram 1. In this diagram, elements

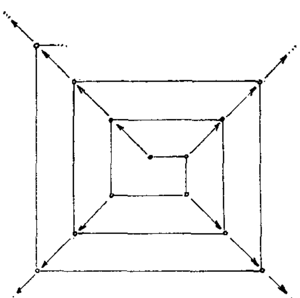

DIAGRAM 1

in the same row are $\mathscr{R}$-related, elements in the same column are $\mathscr{L}$-related and the natural order proceeds down the four diagonals (away from the center). The principal ideals $\omega(e)$ (for $e \in E_4$) are all *ω-chains* (i.e. chains order-isomorphic to the negative integers). We refer to a biordered set in which all principal are ω-chains as an *ω-principal biordered set*: a regular semigroup whose biordered set is ω-principal is called an *ω-principal semigroup.* It was shown in [1] that any *BIG* ω-principal semigroup must contain a copy of Sp_4 as a subsemigroup: however, not every idempotent of a *BIG* ω-principal semigroup need be contained in a copy of E_4. In section 4 we find a class of "building blocks" for *BIG* first that in [11] Nambooripad introduced a *natural partial order* (which we shall denote by $\leqslant$) on a regular semigroup S by defining (for $a, b \in S$) $a \leqslant b$ iff $R_a \leqslant R_b$ and $a = eb$ for some $e = e^2 \in R_a$. There are several equivalent definitions of this partial order relation (see Nambooripad [11] for a discussion of this).

THEOREM 1.6 (Nambooripad [11, 12]). *The following conditions on a regular semigroup S are equivalent:*

(a) *S is pseudo-inverse.*

(b) *For all $e = e^2 \in S$, eSe is an inverse subsemigroup of S.*

(c) *For all $e = e^2 \in E(S)$, $\omega(e)$ is a semilattice.*

(d) *The natural partial order $\leqslant$ on S is compatible with the multiplication in S.*

(e) *For all $e, f \in E(S)$, $|S(e, f)| = 1$.*

(f) *S satisfies the following condition and its dual: if e, f, $g \in E(S)$ with $f\mathscr{R}g$ and $f, g \in \omega(e)$, then $f = g$.*

It is clear from part (c) of the above theorem that ω-principal semigroups are pseudo-inverse.

The natural partial order on a regular semigroup S is closely related to the multiplication in S: it may be used to reduce all products in S to products in the trace of S (see Nambooripad [11]). This partial order is also well-behaved relative to homomorphisms as the following theorem of Nambooripad shows.

THEOREM 1.7 (Nambooripad [11]). *Let ϕ be a homomorphism from a regular semigroup S onto a (regular) semigroup T. Then we have the following.*

(a) *If $a \leqslant b$ in S then $a\phi \leqslant b\phi$ in T.*

(b) *If $c \leqslant d$ in T and b is any element of S for which $b\phi = d$, then there exists an element $a \in S$ such that $a \leqslant b$ and $a\phi = c$.*

(c) *Let $\rho = \phi \circ \phi^{-1}$: then the restriction of the natural partial order on S to each ρ-class $e\rho$ for $e = e^2 \in S$ is trivial if and only if each ρ-class $e\rho$ is a completely simple subsemigroup of S: in particular S is completely simple if and only if the natural partial order relation on S is trivial.*

2. TWO CONSTRUCTIONS OF BIORDERED SETS

In this section we shall introduce two constructions of partially associative pseudo-semilattices. Since a partially associative pseudo-semilattice can be given the structure of a biordered set by defining τ-mappings in the obvious way (Theorem 1.5), each of the constructions produces a class of biordered sets.

The first construction produces biordered sets $P_n^{\infty}(D)$ from the biordered sets $D \cup \{0\}$ of 0-bisimple pseudo-inverse semigroups. We remark that such biordered sets $D \cup \{0\}$ abound; in fact, 0-bisimple inverse semigroups, completely 0-simple semigroups, and normal bands which have a zero and for which the structure semilattice is 0-uniform all have biordered sets $D \cup \{0\}$ of the required type. The second construction produces biordered sets M from families of semilattices and semilattice monomorphisms.

As a preliminary step to the construction of $P_n^{\infty}(D)$ we introduce the biordered set E_n^{∞} called the n-spiral with zero (the zero is denoted by ∞). Let $n > 2$ be an even natural number. Let $E_n = N = \{0, 1, 2, \ldots\}$, $E_n^{\infty} = E_n \cup \{\infty\}$ and define relations ω, ω^l, ω^r on E_n^{∞} as follows:

$$\begin{aligned}
\omega(k) &= \{k + in \mid i \in N\} \cup \{\infty\} \text{ for all } k \in N \\
\omega^r(2k) &= \omega^r(2k+1) = \omega(2k) \cup \omega(2k+1) \text{ for all } k \in N \\
\omega^l(2k) &= \omega^l(2k-1) = \omega(2k) \cup \omega(2k-1) \text{ for all } k \in Z^+ \\
\omega^l(0) &= \omega^l(n) \cup \{0\} \\
\omega^r(\infty) &= \omega^l(\infty) = \omega(\infty) = \{\infty\}.
\end{aligned}$$

It follows immediately from the definitions that ω is a partial order, ω^l and ω^r are quasi-orders and $\omega = \omega^l \cap \omega^r$. The pre-images of elements of $Z_n = Z/(nZ)$ under the obvious mapping $E_n \to Z_n$ are called the *fibres* of E_n and the fibre containing $u \in E_n$ is denoted by f_u. Note that there cannot be both ω^r and ω^l relations between two distinct fibres (i.e. if $x\omega^r y$, $u\omega^l v$, $f_x = f_u$ and $f_y = f_v$ *then* $f_x = f_y$).

Let $\mathscr{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathscr{L} = \omega^l \cap (\omega^l)^{-1}$. Then $\mathscr{R}$ is the equivalence relation on E_n^{∞} with equivalence classes $\{\infty\}$, $\{2k, 2k+1\}$ for $k \in N$ and $\mathscr{L}$ is the equivalence relation on E_n with equivalence classses $\{\infty\}$, $\{0\}$, $\{2k, 2k-1\}$ for $k \in Z^+$. Hence if $x, y \in E_n$ such that $x\omega^r y$, then exactly one of the two elements of R_x, say u, belongs to $\omega(y)$: define $y \wedge x = u$, $x \wedge y = x$. Similarly, if x, $y \in E_n$ such that $x\omega^l y$, then exactly one of the two elements of L_x (or the single element of L_x if $x = y = 0$), say v, belongs to $\omega(y)$: define $x \wedge y = v$, $y \wedge x = x$.

LEMMA 2.1. *$(E_n^{\infty}, \omega^l, \omega^r)$ is a pseudo-semilattice.*

Proof. Let $x, y \in E_n^{\infty}$. We show that there exists a unique $z \in E_n^{\infty}$ such that $\omega^l(x) \cap \omega^r(y) = \omega(z)$. This is clearly true if x or y equals ∞. So suppose $x, y \in E_n$. Then $\omega^l(x) \cap \omega^r(y)$ is clearly an ω-ideal of E_n^{∞}. Furthermore, the non-∞ elements of $\omega^l(x) \cap \omega^r(y)$ are all contained in a single fibre of E_n. For if u, $v \in \omega^l(x) \cap \omega^r(y)$ and if u, $v \neq \infty$, then $u \wedge x \in \omega(x) \wedge L_u$, $v \wedge x \in \omega(x) \cap L_v$; without loss of generality suppose $u \wedge x \leqslant v \wedge x$ in N. Then $v\mathscr{L}v \wedge x\omega u \wedge x\mathscr{L}u$, giving an ω^l relation between f_u and f_v. Similarly there is an ω^r relation between f_u and f_v implying $f_u = f_v$. Thus $\omega^l(x) \cap \omega^r(y)$ is an ideal of an ω-chain with ∞, so there exists a unique $z \in E_n^{\infty}$ such that $\omega^l(x) \cap \omega^r(y) = \omega(z)$.

In the special cases $x\omega^r y$ or $x\omega^l y$ the element z found above is $x \wedge y$. We extend the definition of $\wedge$ by using $x \wedge y$ to denote the element z of the above proof. Thus $\omega^l(x) \cap \omega^r(y) = \omega(x \wedge y)$ for all $x, y \in E_n^\infty$.

LEMMA 2.2. *$(E_n^\infty, \omega^l, \omega^r)$ is a partially associative pseudo-semilattice.*

Proof. We have noted above that $x\omega^r y$ implies $x \wedge y = x$, which is (PA1). Suppose that $y, z \in \omega^r(x)$. Then $x \wedge z \in R_z \cap \omega(x)$, $x \wedge z \mathscr{R} z$ and thus $\omega(y \wedge (x \wedge z)) = \omega^l(y) \cap \omega^r(x \wedge z) = \omega^l(y) \cap \omega^r(z) = \omega(y \wedge z)$, which proves (PA2)(i). Since both $x \wedge z$ and $x \wedge y$ are in $\omega(x)$ we either have (a) $x \wedge z \;\omega\; x \wedge y$ or (b) $x \wedge y \;\omega\; x \wedge z$. If (a) holds, then $z \mathscr{R} x \wedge z \;\omega\; x \wedge y \mathscr{R} y$ so $z\omega^r y$ and thus $y \wedge z \mathscr{R} z$. Therefore $\omega(x \wedge (y \wedge z)) = \omega^l(x) \cap \omega^r(y \wedge z) = \omega^l(x) \cap \omega^r(z) = \omega(x \wedge z)$ so $x \wedge (y \wedge z) = x \wedge z$. On the other hand $\{(x \wedge y) \wedge z\} = R_z \cap \omega(x \wedge y) = R_{x \wedge z} \cap \omega(x \wedge y) = \{x \wedge z\}$, and we conclude $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ which is (PA2)(*ii*). If (*b*) holds, then $x \wedge y \mathscr{R} y \;\omega^r\; z$ so $y \wedge z = y$ and $(x \wedge y) \wedge z = x \wedge y$, so again $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ as required.

By defining τ-mappings appropriately the partially associative pseudo-semilattice $(E_n^\infty, \omega^l, \omega^r)$ may be given the structure of a biordered set $(E_n^\infty, \omega^r, \omega^l, \tau)$, called the *n-spiral with zero*. The n-spiral with 0 is pictured in Diagram 2. We have proved the following theorem.

THEOREM 2.3. *The n-spiral with zero, $E_n^\infty = (E_n^\infty, \omega^r, \omega^l, \tau)$ is the biordered set of a pseudo-inverse semigroup.*

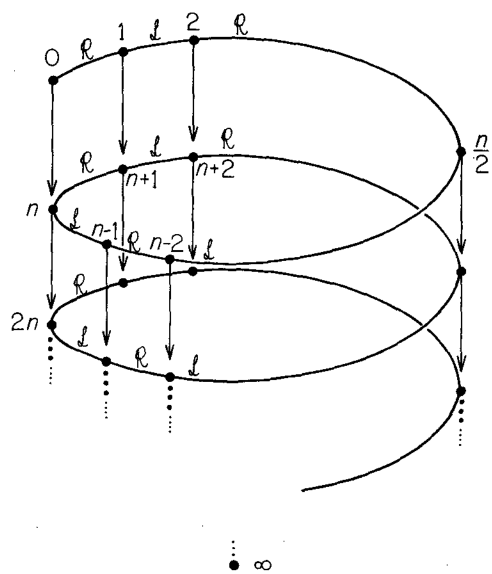

DIAGRAM 2

Remark. If S is a regular IG semigroup having E_n^∞ as its biordered set, then S is 0-bisimple. This follows since any two elements of E_n are linked by an E-chain and since any element of S may be written as a product of the idempotents in an E-chain (see Fitz-Gerald [3] or Nambooripad [10] or Pastijn [13] for a proof of this). Furthermore, since the only E-cycles in E_n^∞ are the trivial E-cycles it follows that the only proper set of E-cycles in E_n^∞ is the set of trivial E-cycles and so by the results of Nambooripad [10, Chap. 6] it follows that the fundamental and the free regular IG semigroups on E_n^∞ coincide and hence there is, up to isomorphism, a unique regular IG semigroup having E_n^∞ as its biordered set.

LEMMA 2.4. *Let $x, y \in E_n^\infty$. Then either* (i) $x \wedge y = \infty$, *or* (ii) $x \wedge y \mathscr{L} x$, *or* (iii) $x \wedge y \mathscr{R} y$.

Proof. Suppose $x \wedge y \neq \infty$ and let u be the maximal element of the fibre $f_{x \wedge y}$. By the maximality of u we have either (1) $x \omega^l u$ and $y \omega^r u$ or (2) $x = 0$, $u = n - 1$ and $y \omega^r u$. In the first case $x \wedge u \in L_x \cap \omega(u)$, in the second case $x \wedge u = u$ and in both cases $u \wedge y \in R_y \cap \omega(u)$, so $x \wedge u$ and $u \wedge y$ are ω-comparable and the smaller is easily seen to be $x \wedge y$.

Let $(D \cup \{0\}, \omega^r, \omega^l, \tau)$ be the biordered set of a 0-bisimple pseudo-inverse semigroup and let $a \in D$. Then $\omega(a)$ is a semilattice with zero 0. Let $\omega(a)^*$ denote $\omega(a) \backslash \{0\}$. Let $n > 2$ be any even natural number and let $P_n(D) = (D \times \{0\}) \cup (\omega(a)^* \times (E_n \backslash \{0\}))$, $P_n^\infty(D) = P_n(D) \cup \{\infty\}$. Thus $P_n(D) \subset D \times E_n$. Define quasi-orders ω^r and ω^l on $P_n(D)$ as follows:

$$(e, x)\ \omega^r\ (f, y) \text{ if either (1) } x \mathscr{R} y \text{ and } e\ \omega^r f, \text{ or (2) } x\ \omega^r y,\ (x, y) \notin \mathscr{R}.$$
$$(e, x)\ \omega^l\ (f, y) \text{ if either (1) } x \mathscr{L} y \text{ and } e\ \omega^l f, \text{ or (2) } x\ \omega^l y,\ (x, y) \notin \mathscr{L}.$$

We extend these quasi-orders to $P_n^\infty(D)$ in the obvious way. Note that three sets of ω^r, ω^l relations appear in the above definitions: those from E_n^∞, those from $D \cup \{0\}$ and those we are defining. The choice of symbols for elements of the respective sets indicate which quasi-order is intended. We remark that $e\ \omega^r f$ iff $e\ \omega^l f$ iff $e\ \omega f$ for $e, f \in \omega(a)^*$.

LEMMA 2.5. $(P_n^\infty(D), \omega^l, \omega^r)$ *is a pseudo-semilattice.*

Proof. Let $(e, x), (f, y) \in P_n^\infty(D)$ and define the operation $\wedge$ as follows (by Lemma 2.4 all cases are covered):

(i) $(e, x) \wedge (f, y) = \infty$ if $x \wedge y = \infty$.

(ii) $(e, x) \wedge (f, y) = (f, x \wedge y)$ if $x \wedge y \neq \infty$, $(x \wedge y, x) \notin \mathscr{L}$.

(iii) $(e, x) \wedge (f, y) = (e, x \wedge y)$ if $x \wedge y \neq \infty$, $(x \wedge y, y) \notin \mathscr{R}$.

(iv) $(e, x) \wedge (f, y) = (e \wedge f, x \wedge y)$ if $x \wedge y \neq \infty$, $x \mathscr{L} x \wedge y \mathscr{R} y$, $(e \wedge f, x \wedge y) \in P_n(D)$.

(v) $(e, x) \wedge (f, y) = (a, (x \wedge y) + n)$ if $x \wedge y \neq \infty$, $x \mathscr{L} x \wedge y \mathscr{R} y$, $(e \wedge f, x \wedge y) \notin P_n(D)$.

(vi) $\infty \wedge (e, x) = (e, x) \wedge \infty = \infty$.

We claim that $\omega^l(e, x) \cap \omega^r(f, y) = \omega((e, x) \wedge (f, y))$ in each instance (we write $\omega^l(e, x)$ instead of $\omega^l((e, x))$). This is obvious for (i) and (vi).

(ii) Suppose $x \wedge y \neq \infty$, $(x \wedge y, x) \notin \mathscr{L}$. Then by Lemma 2.4 we have $x \wedge y \mathscr{R} y$. Hence $(f, x \wedge y) \mathscr{R} (f, y)$ and $(f, x \wedge y) \omega^l (e, x)$, so $(f, x \wedge y) \in \omega^l(e, x) \cap \omega^r(f, y)$. Suppose $(g, z) \in \omega^l(e, x) \cap \omega^r(f, y)$. Then $z \in \omega^l(x) \cap \omega^r(y)$ so $z \,\omega\, x \wedge y$. If $z = x \wedge y$, then $(g, z) \,\omega^r (f, y)$ implies $g \,\omega\, f$, so $(g, z) \,\omega\, (f, x \wedge y)$. If $z \neq x \wedge y$, then since $z \,\omega\, x \wedge y$ we have $(g, z) \,\omega\, (f, x \wedge y)$. Hence $\omega(f, x \wedge y) = \omega^l(e, x) \cap \omega^r(f, y)$ as required. Case (iii) is similar to (ii).

(iv) and (v). Suppose $x \wedge y \neq \infty$, $x \mathscr{L} x \wedge y \mathscr{R} y$. If $(e \wedge f, x \wedge y) \in P_n(D)$, then clearly $\omega(e \wedge f, x \wedge y) = \omega^l(e, x) \cap \omega^r(f, y)$. If $(e \wedge f, x \wedge y) \notin P_n(D)$, then either (a) $e \wedge f = 0$ or (b) $e \wedge f \in D \backslash \omega(a)^*$ with $x \wedge y \neq 0$. If (a) holds, then $(a, (x \wedge y) + n) \in \omega^l(e, x) \cap \omega^r(f, y)$ and if $(g, z) \in \omega^l(e, x) \cap \omega^r(f, y)$, then $z \,\omega\, x \wedge y$. But $e \wedge f = 0$ so $z \neq x \wedge y$, hence $(g, z) \,\omega\, (a, (x \wedge y) + n)$. Thus $\omega^l(e, x) \cap \omega^r(f, y) = \omega(a, (x \wedge y) + n)$. If (*b*) holds, then $(a, (x \wedge y) + n) \in \omega^l(e, x) \cap \omega^r(f, y)$ and if $(g, z) \in \omega^l(e, x) \cap \omega^r(f, y)$, then $z \,\omega\, x \wedge y$. But $z = x \wedge y \neq 0$ is impossible, since $e \wedge f \notin \omega(a)^*$. Hence $z \neq x \wedge y$ so $(g, z) \,\omega\, (a, (x \wedge y) + n)$.

LEMMA 2.6. *$(P_n^{\infty}(D), \omega^l, \omega^r)$ is a partially associative pseudo-semilattice.*

Proof. Since (PA1) is obvious if ∞ is one of the two elements we suppose $(e, x) \,\omega^r (f, y)$. Then either (1) $x \mathscr{R} y$ and $e \,\omega^r f$ or (2) $x \,\omega^r y$, $(x, y) \notin \mathscr{R}$. In each case $x \wedge y = x$. If (1) holds, then $e \wedge f = e$ so either (iii) or (iv) is used to define $(e, x) \wedge (f, y)$ which thus equals (e, x) as required. If (2) holds then (iii) yields $(e, x) \wedge (f, y) = (e, x)$.

Since (PA2) is obvious if ∞ is one of the elements we suppose $(f, y), (g, z) \in \omega^r(e, x)$. If $z \mathscr{R} x$, then $g \,\omega^r e$, so $x \wedge z \mathscr{R} z$ and $g \mathscr{R} e \wedge g \,\omega\, e$. Thus $(e, x) \wedge (g, z) \mathscr{R} (g, z)$. If $(z, x) \notin \mathscr{R}$, then since $z \mathscr{R} x \wedge z \,\omega\, x$ we have $(x \wedge z, x) \notin \mathscr{L}$. Thus $(e, x) \wedge (g, z)$ is defined by (ii), so equals $(g, x \wedge z)$ which implies $(e, x) \wedge (g, z) \mathscr{R} (g, z)$. It follows that $(f, y) \wedge ((e, x) \wedge (g, z)) = (f, y) \wedge (g, z)$ which is (PA2)(i). To prove (PA2)(ii) we note that since $y, z \in \omega^r(x)$ we have that $x \wedge z$ and $x \wedge y$ are ω-comparable since both are in $\omega(x)$. By considering the six possible orderings of $x \wedge z$, $x \wedge y$ and x we may routinely compute the products necessary to verify (PA2)(ii); we have the following cases: (1) $x \wedge z \neq^{\omega} x \wedge y \neq^{\omega} x$, (2) $x \wedge y \neq^{\omega} x \wedge z \neq^{\omega} x$, (3) $x \wedge z \neq^{\omega} x \wedge y = x$, (4) $x \wedge y \neq^{\omega} x \wedge z = x$, (5) $x \wedge y = x \wedge z \neq^{\omega} x$, (6) $x \wedge y = x \wedge z = x$. To illustrate the computation we consider (1). Then by the definition of $\wedge$ we have $(f, y) \wedge (g, z) = (g, y \wedge z)$, $(e, x) \wedge (g, y \wedge z) = (g, x \wedge (y \wedge z))$, $(e, x) \wedge (f, y) =$

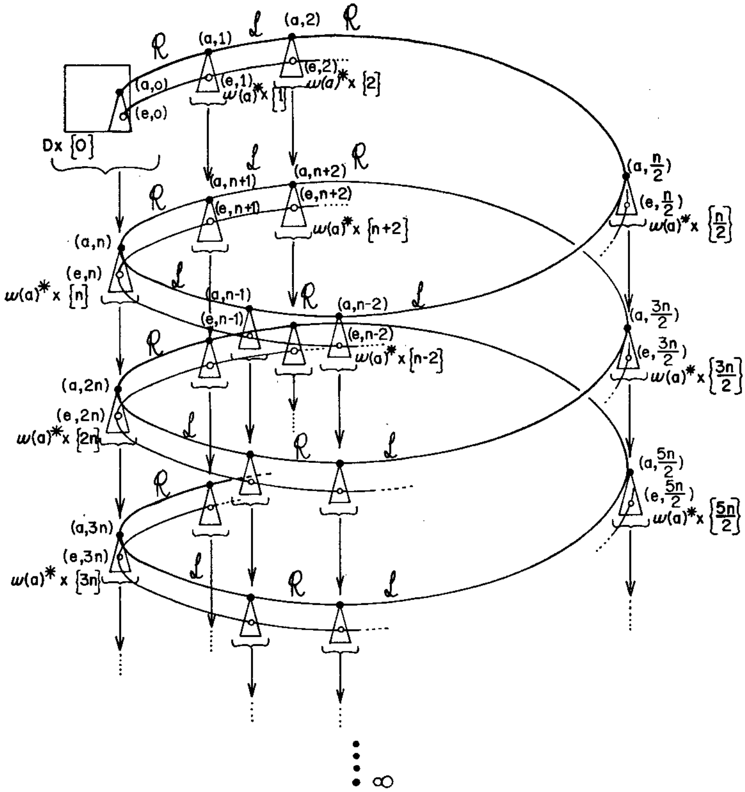

DIAGRAM 3

$(f, x \wedge y)$ and $(f, x \wedge y) \wedge (g, z) = (g, (x \wedge y) \wedge z)$ which imply $(e, x) \wedge ((f, y) \wedge (g, z)) = ((e, x) \wedge (f, y)) \wedge (g, z)$ by the fact that E_n^∞ satisfies (PA2)(ii).

The procedure for manufacturing a biordered set from a partially associative pseudo-semilattice proves the following theorem. The biordered set $P_n^\infty(D)$ is pictured in Diagram 3.

THEOREM 2.7. *$P_n^\infty(D) = (P_n^\infty(D), \omega^e, \omega^l, \tau)$ is the biordered set of a pseudo-inverse semigroup.*

We remark that the principal ideals of elements other than ∞ of $P_n^\infty(D)$ are all isomorphic to an ω-chain of copies of $\omega(a)^*$ together with a zero ∞. Furthermore, $P_n(D)$ is connected if and only if D is connected.

We next construct a second class of biordered sets which are pseudo-semilattices. If E is a semilattice we denote the principal ideal generated by $x \in E$ by $\langle x \rangle$. Let I, Λ be index sets and let $\{R_i \mid i \in I\}$, $\{L_\lambda \mid \lambda \in \Lambda\}$ and $\{M_{i\lambda} \mid (i, \lambda) \in I \times \Lambda\}$ be indexed families of semilattices such that the $M_{i\lambda}$ are pairwise disjoint.

Let $\psi_{i\lambda}: M_{i\lambda} \to R_i$ and $\phi_{i\lambda}: M_{i\lambda} \to L_\lambda$ be monomorphisms such that for all $(i, \lambda) \in I \times \Lambda$

(M1) $M_{i\lambda}\psi_{i\lambda}$ and $M_{i\lambda}\phi_{i\lambda}$ are ideals of R_i and L_λ respectively;

(M2) $R_i = \bigcup_{\lambda\in\Lambda} M_{i\lambda}\psi_{i\lambda}$, $L_\lambda = \bigcup_{i\in I} M_{i\lambda}\phi_{i\lambda}$;

(M2) for every $x \in R_i$, $y \in L_\lambda$ there exists some $z \in M_{i\lambda}$ such that $\langle z\rangle = (\langle x\rangle \cap M_{i\lambda}\psi_{i\lambda})\psi_{i\lambda}^{-1} \cap (\langle y\rangle \cap M_{i\lambda}\phi_{i\lambda})\phi_{i\lambda}^{-1}$.

Let $M = \bigcup_{i,\lambda} M_{i\lambda}$ and define relations ω^r and ω^l on M as follows: let $x, y \in M$, say $x \in M_{i\lambda}$, $y \in M_{j\mu}$;

$$x\ \omega^r\ y \text{ if } i = j \text{ and } x\psi_{i\lambda} \leqslant y\psi_{j\mu} \text{ in } R_i,$$
$$x\ \omega^l\ y \text{ if } \lambda = \mu \text{ and } x\phi_{i\lambda} \leqslant y\phi_{j\mu} \text{ in } L_\lambda.$$

It is clear that the relations ω^r and ω^l are quasi-orders on M. Let $\mathscr{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathscr{L} = \omega^l \cap (\omega^l)^{-1}$, $\omega = \omega^r \cap \omega^l$. By (M3) we may define a binary operation $\wedge$ on M by the following: if $x \in M_{i\lambda}$, $y \in M_{j\mu}$, then

$$(*) \qquad \langle x \wedge y\rangle = (\langle x\phi_{i\lambda}\rangle \cap M_{j\lambda}\phi_{j\lambda})\phi_{j\lambda}^{-1} \cap (\langle y\psi_{j\mu}\rangle \cap M_{j\lambda}\psi_{j\lambda})\psi_{j\lambda}^{-1}.$$

Obviously $x \wedge y \in M_{j\lambda}$.

LEMMA 2.8. *(M, ω^l, ω^r) is a pseudo-semilattice.*

Proof. We must show that $\omega^l(x) \cap \omega^r(y) = \omega(x \wedge y)$. Let $x \in M_{i\lambda}$, $y \in M_{j\mu}$ and suppose $z \in \omega^l(x) \cap \omega^r(y)$. Then $z \in M_{j\lambda}$, $z\phi_{j\lambda} \leqslant x\phi_{i\lambda}$ and $z\psi_{j\lambda} \leqslant y\psi_{j\mu}$ in L_λ and R_j respectively. Consequently $z \in (\langle x\phi_{i\lambda}\rangle \cap M_{j\lambda}\phi_{j\lambda})\phi_{j\lambda}^{-1} \cap (\langle y\psi_{j\mu}\rangle \cap M_{j\lambda}\psi_{j\lambda})\psi_{j\lambda}^{-1} = \langle x \wedge y\rangle$. On the other hand, if $z \in \omega(x \wedge y)$, then $z \leqslant x \wedge y$ in $M_{j\lambda}$ and so $z\phi_{j\lambda} \in \langle x\phi_{i\lambda}\rangle$ in L_λ and $z\psi_{j\lambda} \in \langle y\psi_{j\mu}\rangle$ in R_j. Thus $z \in \omega^l(x) \cap \omega^r(y)$. We conclude $\omega^l(x) \cap \omega^r(y) = \omega(x \wedge y)$.

LEMMA 2.9. *(M, ω^l, ω^r) is a partially associative pseudo-semilattice.*

Proof. We must verify (PA1) and (PA2). Note first that for any $x \in M$, say $x \in M_{i\lambda}$, we have $\langle x\rangle\psi_{i\lambda} = \langle x\psi_{i\lambda}\rangle$ since $M_{i\lambda}\psi_{i\lambda}$ is an ideal of R_i. Suppose now that $x, y, z \in M$, say $x \in M_{i\lambda}$, $y \in M_{j\mu}$, $z \in M_{k\nu}$.

(PA1) Suppose $x\ \omega^r\ y$. Then $i = j$ and $x\psi_{i\lambda} \leqslant y\psi_{j\mu}$ in R_i. Hence by (*) $\langle x \wedge y\rangle = \langle x\rangle \cap (\langle y\phi_{j\mu}\rangle \cap M_{j\lambda}\phi_{j\lambda})\psi_{j\lambda}^{-1} = \langle x\rangle$, so $x \wedge y = x$.

(PA2) Suppose $y, z \in \omega^r(x)$. Then $i = j = k$ and $y\psi_{j\mu} \leqslant x\psi_{i\lambda}$ in R_i, $z\psi_{k\nu} \leqslant x\psi_{i\lambda}$ in R_i. Using (*) to compute we obtain the following equations:

(1) $\langle x \wedge y\rangle\,\psi_{i\lambda} = \langle x\psi_{i\lambda}\rangle \cap \langle y\psi_{j\mu}\rangle = \langle y\psi_{j\mu}\rangle$

(2) $\langle x \wedge z\rangle\,\psi_{i\lambda} = \langle z\psi_{k\nu}\rangle$

(3) $\langle y \wedge z\rangle \psi_{j\mu} = \langle y\psi_{j\mu}\rangle \cap \langle z\psi_{k\nu}\rangle$

(4) $\langle (x \wedge y) \wedge z\rangle \psi_{i\lambda} = \langle x \wedge y\rangle \psi_{i\lambda} \cap \langle z\psi_{k\nu}\rangle = \langle y\psi_{j\mu}\rangle \cap \langle z\psi_{k\nu}\rangle$

(5) $\langle x \wedge (y \wedge z)\rangle \psi_{i\lambda} = \langle x\rangle \psi_{i\lambda} \cap \langle (y \wedge z) \psi_{j\mu}\rangle = \langle (y \wedge z) \psi_{j\mu}\rangle$.

From (2) we have $x \wedge z \mathscr{R} z$. Hence $y \wedge (x \wedge z) = y \wedge z$, which is (PA2)(i). From (3), (4) and (5) it follows that $(x \wedge y) \wedge z \mathscr{R} x \wedge (y \wedge z)$. But $x \wedge (y \wedge z) \omega^r y \wedge z \omega^r z$ and $x \wedge (y \wedge z) \in \omega^l(x) \cap \omega^r(y \wedge z) \subseteq \omega^l(x) \cap \omega^r(y) = \omega(x \wedge y)$, so $x \wedge (y \wedge z) \omega (x \wedge y) \wedge z$. Hence $((x \wedge y) \wedge z) \omega^r \cap (\omega^l)^{-1}(x \wedge (y \wedge z))$ which by (PA1) implies $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, which is (PA2)(ii).

Again the method of producing a biordered set from a partially associative pseudo-semilattice yields the following result:

THEOREM 2.10. *$M = (M, \omega^r, \omega^l, \tau)$ is the biordered set of a pseudo-inverse semigroup.*

We say that the biordered set B is a rectangular biordered set if for all $e, f \in B$ there exist $g, h \in B$ such that $e \mathscr{R} g \mathscr{L} f \mathscr{R} h \mathscr{L} e$: the biordered set E is a coextension of a rectangular biordered set by semilattices if there is a bimorphism θ from E onto a rectangular biordered set B such that for all $\alpha \in B$, $\alpha(\theta \circ \theta^{-1})$ is a semilattice.

We remark that the construction of Theorem 2.10 produces all biordered sets which are coextensions of a rectangular biordered set by semilattices. Certainly the map $\phi: M \to I \times \Lambda$ defined by $x \mapsto (i, \lambda)$ for $x \in M_{i\lambda}$ is a bimorphism from M onto the rectangular biordered set $I \times \Lambda$ with the $M_{i\lambda}$ as $\phi \circ \phi^{-1}$ classes, so M is a coextension of $I \times \Lambda$ by semilattices. Conversely, suppose the biordered set E is a coextension of the rectangular biordered set B by semilattices, say $\phi: E \twoheadrightarrow B$ where each $e(\phi \circ \phi^{-1})$ is a semilattice. Then each $e(\phi \circ \phi^{-1})$ is a maximal subsemilattice of E since if $e\phi \neq f\phi$, then $e\phi$ and $f\phi$, and hence e and f, do not commute. Thus E is the disjoint union of its maximal subsemilattices. Pastijn [14] and [16] has shown that any biordered set which is the disjoint union of its maximal subsemilattices is biorder isomorphic to some M of Theorem 2.10. This comment applies in particular to rectangular bands of inverse semigroups and to pseudo-inverse semigroups with one-to-one structure mappings (see Nambooripad [9] or Meakin [5] for a description of the structure mappings), both of which have biordered sets which are the disjoint unions of their maximal subsemilattices [14].

3. COUNTEREXAMPLES TO THE EMBEDDING QUESTION

Each of the two constructions of biordered sets of the previous section may be used to build counterexamples to the embedding question, that is, to construct bisimple idempotent-generated semigroups which are not completely simple and do not contain copies of the fundamental four-spiral semigroup Sp_4.

EXAMPLE 3.1. Let $n > 4$ be an even natural number. We define a sequence $Q_1^\infty, Q_2^\infty, Q_3^\infty,\ldots$ of biordered sets of pseudo-inverse semigroups by repeated use of Theorem 2.7: $Q_1^\infty = E_n^\infty$ (the n-spiral with zero), $Q_2^\infty = P_n^\infty(Q_1),\ldots, Q_k^\infty = P_n^\infty(Q_{k-1}),\ldots$. Since the mapping $x \mapsto (x, 0)$, $x \in D$, $0 \mapsto (a, 1)$ is a biorder isomorphism of $D \cup \{0\}$ into $P_n^\infty(D)$ we may consider $Q_1^\infty \subsetneq Q_2^\infty \subsetneq \cdots$. Let $Q = \bigcup_{k=1}^{\infty} Q_k^\infty$. We define $e\,\omega^r f$ in Q if $e\,\omega^r f$ in some Q_k^∞ and similarly define ω^l in Q. Then Q is easily checked to be a partially associative pseudo-semilattice and thus Q may be given the structure of a biordered set. Since for any $k > 0$, any two elements of Q_k are linked by an E-chain, Q must be connected, and thus Q is the biordered set of a bisimple idempotent-generated semigroup. Principal ideals of Q are chains of order type ω^ω. The minimum distance between comparable idempotents of Q is n. Hence for even $n > 4$ there exists a bisimple idempotent-generated semigroup which is not completely simple but fails to contain a copy of Sp_4. The biordered sets $Q_1^\infty, Q_2^\infty, Q_3^\infty$ for $n = 6$ are pictured in Diagram 4.

We remark that the construction of Q may be modified in several ways. In particular, different values of n may be chosen in constructing the Q_k's in order to build the biordered set of a bisimple idempotent-generated semigroup in

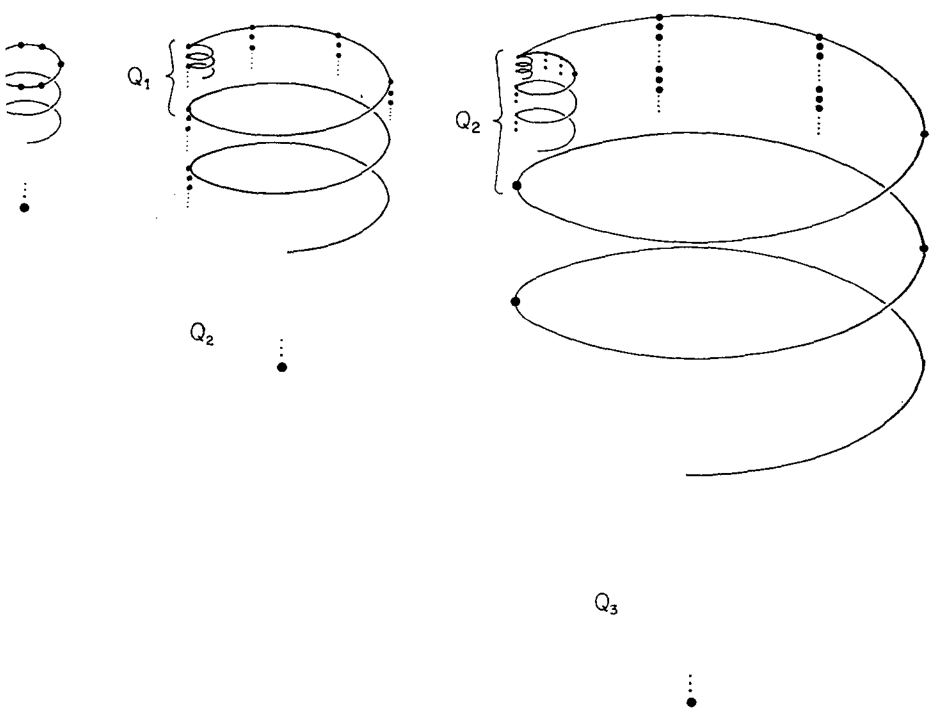

DIAGRAM 4

which the distance between idempotents e, f such that e covers f has no upper bound.

EXAMPLE 3.2. Let $I = \{1, 2\}$, $\Lambda = \{1, 2, 3\}$ and let each of the semilattices R_i, L_λ, $M_{i\lambda}$ for $i \in I$, $\lambda \in \Lambda$ be a copy of the semilattice $(Z \times Z, \leqslant)$ where $(a, b) \leqslant (c, d)$ if $a \geqslant c$ and $b \geqslant d$. Define mappings $\phi_{i\lambda}: M_{i\lambda} \to L_\lambda$ and $\psi_{i\lambda}$: $M_{i\lambda} \to R_i$ as follows:

$$\phi_{i\lambda}: (m, n)_{i\lambda} \mapsto (m, n),\ i \in I,\ \lambda \in \Lambda$$
$$\psi_{1\lambda}: (m, n)_{1\lambda} \mapsto (m, n),\ \lambda \in \Lambda$$
$$\psi_{21}: (m, n)_{21} \mapsto (m, n)$$
$$\psi_{22}: (m, n)_{22} \mapsto (m + 1, n - 1)$$
$$\psi_{23}: (m, n)_{23} \mapsto (m - 1, n + 2).$$

Then conditions (M1), (M2), (M3) are easily seen to be satisfied and so $M = \bigcup M_{i\lambda}$ is the biordered set of a pseudo-inverse semigroup by Theorem 2.10. M is pictured in diagram 5. As indicated by the $\mathscr{R}$ and $\mathscr{L}$ relations in the diagram, every element $(m, n)_{11}$ is linked by an E-chain to both $(m + 1, n)_{11}$ and $(m, n + 1)_{11}$. It follows that any two elements of M_{11} are linked by an E-chain. Since every element of M is linked to an element of M_{11} by an E-chain we conclude that any two elements of M are linked by an E-chain. Hence M is connected and is the biordered set of some bisimple idempotent-generated semigroup S.

Since each $\mathscr{R}$-class of S contains exactly 3 idempotents and each $\mathscr{L}$-class 2 idempotents, the element $(m, n)_{11}$ of M_{11} is linked to just 2 other elements of M_{11} by chains of length 4 which start with an $\mathscr{R}$-relation, namely $(m + 1, n - 1)_{11}$ and $(m - 1, n + 2)_{11}$, neither of which is comparable to $(m, n)_{11}$. We conclude that no element of M_{11} belongs to a four-spiral subsemigroup of S. Similarly, no element of M_{12} belongs to a four-spiral subsemigroup of S. Since any four-spiral subsemigroup of S must contain either an element of M_{11} or an element of M_{12} we conclude that S fails to contain a copy of Sp_4.

4. BUILDING BLOCKS FOR BIG ω-PRINCIPAL SEMIGROUPS

Although every BIG ω-principal semigroup S contains an isomorphic copy of Sp_4 as a subsemigroup, it is not necessarily the case that every $e \in E(S)$ is contained in such a subsemigroup. It is thus of interest to find a family $\mathscr{F}$ of BIG ω-principal semigroups such that for any BIG ω-principal semigroup S and any $e \in E(S)$ there exists an element $F \in \mathscr{F}$ such that e belongs to a subsemigroup of S isomorphic to F. We shall define such a family $\mathscr{F}$ using one of the constructions of pseudo-semilattices of Section 2. First we note some useful facts concerning congruences and homomorphisms on bisimple ω-principal semigroups.

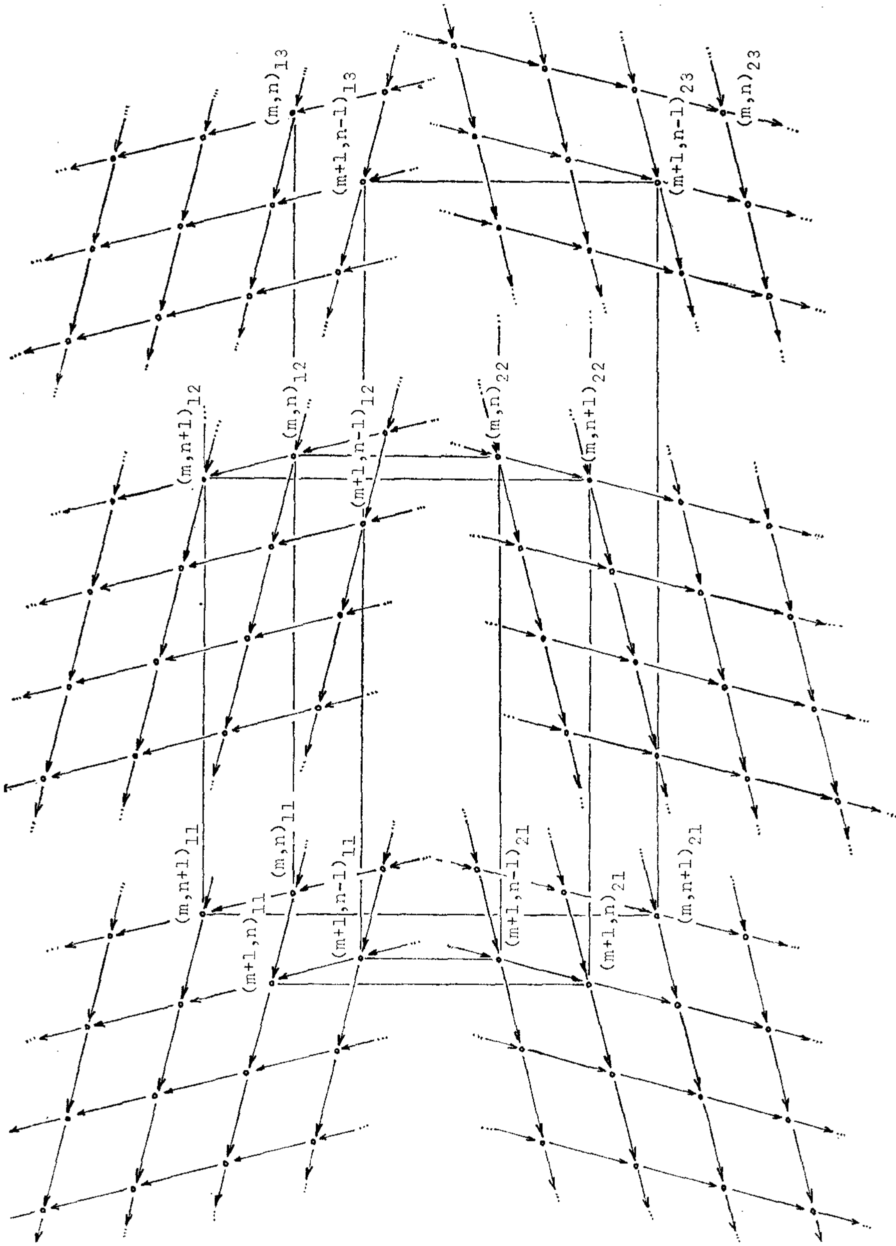
(m,n)13
(m+1,n-1)13
(m+1,n-1)23
(m,n)23
(m,n+1)12
(m,n)12
(m+1,n-1)12
(m,n)22
(m,n+1)22
(m,n+1)11
(m,n)11
(m+1,n-1)11
(m+1,n)11
(m+1,n-1)21
(m+1,n)21
(m,n+1)21

DIAGRAM 5

PROPOSITION 4.1. *Let ρ be a congruence on a bisimple ω-principal semigroup S. Then either S/ρ is completely simple, or $e\rho$ is a completely simple subsemigroup of S for all $e \in E(S)$: in the latter case, S/ρ is also a bisimple ω-principal semigroup.*

Proof. Suppose ρ identifies comparable idempotents e, f, say $f\,\omega\,e$ with $e \neq f$. Since S is bisimple, e and f belong to a bicyclic subsemigroup of S and hence ρ identifies all elements of $\omega(e)$. By Theorem 1.7(b) and the fact that $E(S)$ forms an order-ideal of the partially ordered set $(S, \leqslant)$, it follows that $e\rho$ is primitive so S/ρ is completely simple. If no two comparable idempotents are identified, then $e\rho$ is completely simple $\forall e \in E(S)$ by Theorem 1.7(c). The final statement of the proposition is clear.

Remark. Congruences ρ on a regular semigroup S for which each $e\rho$ ($e \in E(S)$) is a completely simple subsemigroup of S have been briefly discussed by Meakin and Nambooripad [6]: it is easy to see that the set of all such congruences forms a complete sublattice of the lattice of congruences on S.

COROLLARY 4.2. *Let ϕ: $S \to T$ be a homomorphism of the bisimple ω-principal semigroup S onto the bisimple ω-principal semigroup T. Let $e \in E(S)$. Then $\phi|_{\omega(e)}$ is an isomorphism from $\omega(e)$ onto $\omega(e\phi)$.*

Proof. By Theorem 1.7(b), $\phi|_{\omega(e)}$ maps $\omega(e)$ onto $\omega(e\phi)$. Let $\rho = \phi \circ \phi^{-1}$. Since $S/\rho \cong T$ is not completely simple, each $f\rho$, $f \in \omega(e)$, is completely simple and hence $\phi|_{\omega(e)}$ is also one-to-one.

Let S be a bisimple ω-principal semigroup and let $g \in E(S)$. Let γ be the obvious isomorphism from $\omega(g)$ onto $N = \{0, 1, 2, \ldots\}$. If $g' \in \omega(g)$ we denote the non-negative integer $g'\gamma - g\gamma$ by $g' - g$. Now let $e, f \in E(S)$ and consider $S(e, f) = \{h\}$. Then there is a unique $e' \in \omega(e)$ such that $e' \mathscr{L} h$ and there is a unique $f' \in \omega(f)$ such that $f' \mathscr{R} h$. We call the ordered pair of non-negative integers $(l^{ef}, r^{ef}) = (e' - e, f' - f)$ the *orientation of $S(e, f)$ with respect to e and f.*

COROLLARY 4.3. *Let ϕ: $S \to T$ be a homomorphism from a bisimple ω-principal semigroup S onto a bisimple ω-principal semigroup T. Then ϕ preserves orientations of sandwich sets.*

Proof. Let $e, f \in E(S)$ and let (l^{ef}, r^{ef}) be the orientation of $S(e, f) = \{h\}$. Let $e' \in L_h \cap \omega(e)$. Then $e'\phi \in L_{h\phi} \cap \omega(e\phi)$ and thus, since $S(e\phi, f\phi) = \{h\phi\}$, we have $l^{e\phi, f\phi} = e'\phi - e\phi$. But by Corollary 4.2, $\phi|_{\omega(e)}$ is an isomorphism onto $\omega(e\phi)$, hence $e' - e = e'\phi - e\phi$ and we conclude $l^{ef} = l^{e\phi, f\phi}$. Similarly $r^{ef} = r^{e\phi, f\phi}$ as required.

The free regular IG semigroup $\tilde{E}$ on a biordered set E has been studied by Nambooripad [10] and Pastijn [13]. The following result was proved by Nambooripad [10]: we provide a short proof based on the results of Pastijn [13].

LEMMA 4.4. *Let $\theta: E \to S$ be a bimorphism from a biordered set E into the biordered set of a regular semigroup S. Then θ can be extended uniquely to a homomorphism $\tilde{\theta}: \tilde{E} \to S$ where $\tilde{E}$ is the regular free IG-semigroup with respect to the regular biordered set E. Furthermore, $\tilde{E}\tilde{\theta} = \langle E\theta \rangle$.*

Proof. By Theorem 4.14 of [13], $\tilde{E}$ is the semigroup freely generated by the set $Z = \{\tilde{e} \mid e \in E\}$ subject to the defining relations

$$\tilde{e}\tilde{f} = \tilde{f} \quad \text{if} \quad e \mathscr{R} f \text{ in } E$$

$$\tilde{e}\tilde{f} = \tilde{e} \quad \text{if} \quad e \mathscr{L} f \text{ in } E$$

$$\tilde{e}\tilde{f} = \widetilde{eh}\,\widetilde{hf} \quad \text{if} \quad h \in S(e, f) \text{ in } E.$$

Since θ is a bimorphism, $e \mathscr{R} f$ implies $ef = f$ and so $(ef)\theta = e\theta \cdot f\theta = f\theta$, and thus relations of the first kind hold in $E\theta$. Similarly, relations of the second kind hold in $E\theta$. Suppose $h \in S(e, f)$. Then $h\theta \in S(e\theta, f\theta)$, so $e\theta \cdot f\theta = e\theta h\theta \cdot h\theta f\theta = (eh)\,\theta(hf)\theta$ and thus relations of the third kind hold in $E\theta$. Therefore θ can be extended uniquely to a homomorphism $\tilde{\theta}: \tilde{E} \to S$ which maps $\tilde{E}$ onto $\langle E\theta \rangle$.

LEMMA 4.5. *Suppose the biordered sets E, F are pseudo-semilattices. A mapping $\theta: E \to F$ is a bimorphism if and only if $S(e, f)\theta = S(e\theta, f\theta)$ $\forall e, f \in E$.*

Proof. Since sandwich sets in pseudo-semilattices are singletons, the sandwich set condition is clearly satisfied if θ is a bimorphism. Suppose $S(e, f)\theta = S(e\theta, f\theta)$ $\forall e, f \in E$ and let $e\, \omega^r f$. Then $S(e, f) = \{e\}$ so $S(e\theta, f\theta) = \{e\theta\}$ and $e\theta\, \omega^r f\theta$. It follows that $\theta: E \to F$ is a bimorphism.

We shall now consider a special case of the construction of the biordered set $M = \bigcup M_{i\lambda}$ of section 2. Let $I = \Lambda = \{1, 2,..., n\}$ and suppose L_j, R_i for $1 \leqslant i, j \leqslant n$ are isomorphic copies of an ω-chain $0 > 1 > 2 > \cdots$ which we denote by ω. We also suppose that M_{ij} is an ω-chain for $1 \leqslant i, j \leqslant n$ and we write $M_{ij} = \{k_{ij} \mid k \in N\}$, $0_{ij} > 1_{ij} > 2_{ij} > \cdots$. Let $L = (l^{ij})$, $R = (r^{ij})$ be $n \times n$ matrices with entries in N such that $r^{ii} = l^{ii} = 0$ for $1 \leqslant i \leqslant n$. Define mappings $\phi_{ij}: M_{ij} \to \omega$ by $k_{ij} \mapsto k + l^{ij}$, $\psi_{ij}: M_{ij} \to \omega$ by $k_{ij} \mapsto k + r^{ij}$. Then conditions (M1), (M2) and (M3) are trivially satisfied and so $M = \bigcup M_{ij} = (E, n, L, R)$ is a biordered set. In order to obtain an explicit expression for sandwich sets in (E, n, L, R) we consider a more general situation.

Let S be any regular ω-principal semigroup and let $\bar{0}_{ii}$, $i = 1, 2,..., n$ be any (not necessarily distinct) elements of $E(S)$. Denote $S(\bar{0}_{ii}, \bar{0}_{jj})$ by $\bar{0}_{ji}$. Let $\omega(\bar{0}_{ij}) = \{\bar{k}_{ij} \mid k \in N\}$ where $\bar{0}_{ij} > \bar{1}_{ij} > \cdots$.

LEMMA 4.6. *There exist natural numbers $l^{ij}, r^{ij} \in N$, $1 \leqslant i, j \leqslant n$ such that $\bar{k}_{ij}\, \mathscr{R}\, (\overline{k + r^{ij}})_{ii}$ and $\bar{k}_{ij}\, \mathscr{L}\, (\overline{k + l^{ij}})_{jj}$ for all $1 \leqslant i, j \leqslant n$ and all $j \in N$. Moreover $l^{ii} = r^{ii} = 0$ for $1 \leqslant i \leqslant n$.*

Proof. For all $\overline{0}_{ij}$ we have $\overline{0}_{ij}\,\omega^r\,\overline{0}_{ii}$ and so $\overline{0}_{ij}$ is $\mathscr{R}$-related to exactly one element of $\omega(\overline{0}_{ii})$, say $\overline{r^{ij}_{ii}}$ with $r^{ij}\in N$. Analogously $\overline{0}_{ij}\,\omega^l\,\overline{0}_{jj}$ and so $\overline{0}_{ij}$ is $\mathscr{L}$-related to exactly one element $\overline{l^{ij}_{jj}}$ of $\omega(\overline{0}_{jj})$ where $l^{ij}\in N$. The restriction of $\tau^r(\overline{0}_{ij})$ to $\omega(\overline{r^{ij}_{ii}})$ is then an $\mathscr{R}$-class preserving one-to-one mapping of $\omega(\overline{r^{ij}_{ii}})$ onto $\omega(\overline{0}_{ij})$ and thus $\overline{(r^{ij}+k)}_{ii}\,\mathscr{R}\,\bar{k}_{ij}$. Similarly $\bar{k}_{ij}\,\mathscr{L}\,\overline{(k+l^{ij})}_{jj}$. In particular, $l^{ii}=r^{ii}=0$.

LEMMA 4.7. *For all* $1\leqslant i,j\leqslant n$, $k,t\in N$ *we have* $S(\bar{k}_{ii},\bar{t}_{jj})=\{\bar{x}_{ji}\}$ *where* $x=\max(k-l^{ji},t-r^{ji},0)$.

Proof. Let $\{a\}=S(\bar{k}_{ii},\bar{t}_{jj})$. Then $a\,\omega^l\,\bar{k}_{ii}\,\omega\,\overline{0}_{ii}$ and $a\,\omega^r\,\bar{t}_{jj}\,\omega\,\overline{0}_{jj}$ so $a\in\omega^l(\overline{0}_{ii})\cap\omega^r(\overline{0}_{jj})$. Thus by the definition of sandwich set, $a\,\mathscr{R}\,a\overline{0}_{jj}\,\omega^l\,\overline{0}_{ji}\overline{0}_{jj}\,\mathscr{R}\,\overline{0}_{ji}$. But $a\overline{0}_{jj}$, $\overline{0}_{ji}\overline{0}_{jj}$ are both in the chain $\omega(\overline{0}_{jj})$ and hence $a\overline{0}_{jj}\,\omega\,\overline{0}_{ji}\overline{0}_{jj}$. We conclude $a\,\omega^r\,\overline{0}_{ji}$. Similarly $a\,\omega^l\,\overline{0}_{ji}$ and thus $a\in\omega(\overline{0}_{ji})$. Therefore $S(\bar{k}_{ii},\bar{t}_{jj})=\{\bar{x}_{ji}\}$ for some $x\in N$. There are different cases to consider:

(1) $k>l^{ij}$, $t>r^{ji}$; then $\bar{k}_{ii}\,\mathscr{L}\,\overline{(k-l^{ji})}_{ji}$, $\bar{t}_{jj}\,\mathscr{R}\,\overline{(t-r^{ji})}_{ji}$ and thus

$$\begin{aligned} S(\bar{k}_{ii},\bar{t}_{jj}) &= \overline{(t-r^{ji})}_{ji} &&\text{if } t-r^{ji}\geqslant k-l^{ji}\\ &= \overline{(k-l^{ji})}_{ji} &&\text{if } k-l^{ji}\geqslant t-r^{ji}. \end{aligned}$$

(2) $k>l^{ji}$, $t\leqslant r^{ji}$; then $\overline{0}_{ji}\,\omega^r\,\bar{t}_{jj}$, $\bar{k}_{ii}\,\mathscr{L}\,\overline{(k-l^{ji})}_{ji}$ and thus $S(\bar{k}_{ji},\bar{t}_{jj})=\{\overline{(k-l^{ji})}_{ji}\}$.

(3) $k\leqslant l^{ji}$, $t>r^{ji}$; then $\overline{0}_{ji}\,\omega^l\,\bar{k}_{ii}$, $\bar{t}_{jj}\,\mathscr{R}\,\overline{(t-r^{ji})}_{ji}$ and thus $S(\bar{k}_{ii},\bar{t}_{jj})=\{\overline{(t-r^{ji})}_{ji}\}$.

(4) $k\leqslant l^{ji}$, $t\leqslant r^{ji}$; then $\overline{0}_{ji}\in\omega^l(\bar{k}_{ii})\cap\omega^r(\bar{t}_{jj})$ and thus $S(\bar{k}_{ii},\bar{t}_{jj})=\{\overline{0}_{ji}\}$.

We conclude that $S(\bar{k}_{ii},\bar{t}_{jj})=\{\bar{k}_{ji}\}$ where $x=\max\{k-l^{ji},t-r^{ji},0\}$.

We remark that Lemma 4.7 applies in particular to the biordered set (E,n,L,R) which is the biordered set of a regular ω-principal semigroup. Hence in (E,n,L,R) we have $S(k_{ii},t_{jj})=\{x_{ji}\}$ where $x=\max\{k-l^{ji},t-r^{ji},0\}$. We let $\tilde{E}=(\tilde{E},n,L,R)$ denote the free regular IG-semigroup with respect to the biordered set (E,n,L,R). We determine the greatest completely simple homomorphic image of $(\tilde{E},n,L,R)$.

THEOREM 4.8. *Let* $Q=\{q_{ij}\mid i,j=2,\dots,n\}$ *be a set containing* $(n-1)^2$ *different elements and let* G *be the free group generated by the* $(n-1)^2$ *elements of* Q *and let* e *be the identity of* G (*if* $n=1$, $G=\{e\}$). *The Rees matrix semigroup* $\mathscr{M}=\mathscr{M}(G;I,\Lambda;P)$ *where* $I=\Lambda=\{1,2,\dots,n\}$ *and* $P=(p_{\lambda i})$ *is a* $\Lambda\times I$ *matrix with*

$$\begin{aligned} p_{\lambda i} &= e \text{ if } \lambda \text{ or } i \text{ equals } 1\\ &= q_{\lambda i} \text{ otherwise} \end{aligned}$$

is an IG rectangular band of groups which is the greatest completely simple homomorphic image of $(\tilde{E}, n, L, R)$.

Proof. Let $\{e_{ij} \mid i, j = 1,..., n\}$ be a set containing n^2 elements and let D be the semigroup freely generated by the elements e_{ij} subject to the defining relations $e_{ij}e_{ik} = e_{ik}$ and $e_{ij}e_{kj} = e_{ij}$ for $1 \leqslant i, j, k \leqslant n$. Then D is an IG-rectangular band of groups; D is isomorphic to the Rees matrix semigroup mentioned in the theorem of [15]. The mapping θ: $(E, n, L, R) \to D$ defined by $k_{ij} \mapsto e_{ij}$ is a bimorphism so can be extended uniquely by Lemma 4.4 to a homomorphism $\tilde{\theta}$: $(\tilde{E}, n, L, R) \to D$. Hence D is a completely simple homomorphic image of $(\tilde{E}, n, L, R)$. We claim it is the greatest such image. Let τ be any completely simple congruence on $(\tilde{E}, n, L, R)$. Consider the mapping ψ: $D \to \tilde{E}/\tau$ by $e_{ij} \mapsto \bar{0}_{ij}\tau$. Since $\bar{0}_{ij}\ \tau\ k_{ij}$, $\tilde{E}/\tau$ is generated by the $\tilde{0}_{ij}$, $1 \leqslant i, j \leqslant n$. Furthermore, since $\tilde{0}_{ij}\ \omega^r\ \tilde{0}_{ii}$, $\tilde{0}_{ik}\ \omega^r\ \tilde{0}_{ii}$, we have $\tilde{0}_{ij}\tau\ \mathscr{R}\ \tilde{0}_{ii}\tau\ \mathscr{R}\ \tilde{0}_{ik}\tau$ and thus $\tilde{0}_{ij}\tau = (\tilde{0}_{ik}\tau)(\tilde{0}_{ij}\tau)$. Similarly $\tilde{0}_{ij}\tau = (\tilde{0}_{ij}\tau)(\tilde{0}_{kj}\tau)$. Thus the relations on the generators of D hold in $\tilde{E}/\tau$ so ψ can be extended to a homomorphism from D onto $\tilde{E}/\tau$. This proves that D is the greatest completely simple homomorphic image of $(\tilde{E}, n, L, R)$.

Now we introduce a family of semigroups which will serve as building blocks for BIG ω-principal semigroups. Let (B, n, L, R) denote any biordered set (E, n, L, R) defined above in which L and R satisfy the additional restrictions

$$l^{i,i+1} = r^{i,i+1} = 0 \text{ for } 1 \leqslant i \leqslant n-1$$
$$l^{n1} = 1, \qquad r^{n1} = 0.$$

Then $0_{11}\ \mathscr{R}\ 0_{12}\ \mathscr{L}\ 0_{22} \cdots 0_{n-1,n}\ \mathscr{L}\ 0_{nn}\ \mathscr{R}\ 0_{n1}\ \mathscr{L}\ 1_{11}$ is an E-sequence linking 0_{11} with 1_{11}. From this observation it is easy to see that any two elements of (B, n, L, R) are linked by an E-sequence, and thus that (B, n, L, R) is the biordered set of a BIG ω-principal semigroup, for example, of $\tilde{B} = (\tilde{B}, n, L, R)$, the free regular IG-semigroup with respect to (B, n, L, R). Let $\mathscr{F}$ denote the family of all non-completely simple homomorphic images $(\tilde{B}, n, L, R)/\rho$ of the BIG ω-principal semigroups $(\tilde{B}, n, L, R)$. Note that each element of $\mathscr{F}$ is a BIG ω-principal semigroup by Proposition 4.1.

THEOREM 4.9. *$\mathscr{F}$ is a family of building blocks for BIG ω-principal semigroups, i.e. if S is a BIG ω-principal semigroup and $e \in E(S)$, then there is a subsemigroup of S containing e which is isomorphic to some* $(\tilde{B}, n, L, R)/\rho \in \mathscr{F}$.

Proof. Let S be a BIG ω-principal semigroup and let $e \in E(S)$. Suppose that $f\ \omega\ e$, $f \neq e$. Then there is an E-sequence joining e and f of the form $e = \bar{0}_{11}\ \mathscr{R}\ \bar{0}_{12}\ \mathscr{L}\ \bar{0}_{22}\ \mathscr{R} \cdots \mathscr{L}\ \bar{0}_{nn}\ \mathscr{R}\ \bar{0}_{n1}\ \mathscr{L}\ \bar{1}_{11} = f$. We emphasize that the idempotents in this E-sequence are not necessarily distinct. Let the sandwich set

$S(\bar{0}_{ii}, \bar{0}_{jj})$ be denoted by $\bar{0}_{ji}$. Note that this is in agreement with our earlier notation since

$$\begin{aligned}
\{\bar{0}_{ii}\} &= S(\bar{0}_{ii}, \bar{0}_{ii}) \text{ for all } 1 \leqslant i \leqslant n,\\
\{\bar{0}_{i,i+1}\} &= S(\bar{0}_{i+1,i+1}, \bar{0}_{ii}) \text{ for all } 1 \leqslant i \leqslant n-1\\
&\quad \text{since then } \bar{0}_{ii} \mathscr{R} \bar{0}_{i,i+1} \mathscr{L} \bar{0}_{i+1,i+1},\\
\{\bar{0}_{n1}\} &= S(\bar{0}_{11}, \bar{0}_{nn}) \text{ since } \bar{0}_{n1} \omega^l \bar{0}_{11} \text{ and } \bar{0}_{n1} \mathscr{R} \bar{0}_{nn}.
\end{aligned}$$

We let $\omega(\bar{0}_{ii}) = \{\bar{k}_{ij} \mid k \in N\}$ as above.

Now let (l^{ij}, r^{ij}) denote the orientation of the sandwich set $\{\bar{0}_{ij}\}$ with respect to $\bar{0}_{jj}$ and $\bar{0}_{ii}$ and consider the matrices $L = (l^{ij})$ and $R = (e^{ij})$. Then

$$\begin{aligned}
l^{i,i+1} = r^{i,i+1} = 0 \qquad &\text{for} \quad 1 \leqslant i \leqslant n-1\\
l^{n1} = 1, \qquad &r^{n1} = 0
\end{aligned}$$

and thus we have the biordered set (B, n, L, R) of a *BIG* ω-principal semigroup. By Lemmas 4.5 and 4.7 the mapping θ: $(B, n, L, R) \to S$ defined by $k_{ij} \mapsto \bar{k}_{ij}$ is a bimorphism, so by Lemma 4.4 it may be extended uniquely to a homomorphism δ: $(\tilde{B}, n, L, R) \to S$. Hence e belongs to a subsemigroup of S isomorphic to $(\tilde{B}, n, L, R)/\rho$ for some congruence ρ. Since e, $f \in (\tilde{B}, n, L, R)\delta$, $(\tilde{B}, n, L, R)/\rho$ is not completely simple (Proposition 4.1).

Additional information concerning the building blocks for *BIG* ω-principal semigroups is provided by the following theorem.

THEOREM 4.10. *$\{\tilde{0}_{ij} \mid i, j = 1,\dots, n\}$ is a minimal set of idempotent generators for $(\tilde{B}, n, L, R)$.*

Proof. Suppose that $\tilde{k}_{ii}$ can be written as a product of elements $\tilde{0}_{pq}$, $1 \leqslant p$, $q \leqslant n$, for some $k \in N$ and some $1 \leqslant i \leqslant n$. Suppose $i < n$. Since $\tilde{k}_{ii} \,\omega\, \tilde{0}_{ii} \mathscr{R} \tilde{0}_{i,i+1}$ we have $\tilde{k}_{ii}\tilde{0}_{i,i+1} \mathscr{R} \tilde{k}_{ii}$ and $\tilde{k}_{ii}\tilde{0}_{i,i+1} \,\omega\, \tilde{0}_{i,i+1}$ and so $\tilde{k}_{ii}\tilde{0}_{i,i+1} = \tilde{k}_{i,i+1}$. Since $\tilde{k}_{i,i+1} \,\omega\, \tilde{0}_{i,i+1} \mathscr{L} \tilde{0}_{i+1,i+1}$ we have $\tilde{0}_{i+1,i+1}\tilde{k}_{i,i+1} \mathscr{L} \tilde{k}_{i,i+1}$ and $\tilde{0}_{i+1,i+1}\tilde{k}_{i,i+1} \,\omega\, \tilde{0}_{i+1,i+1}$ and so $\tilde{k}_{i+1,i+1} = \tilde{0}_{i+1,i+1}\tilde{k}_{i,i+1} = \tilde{0}_{i+1,i+1}\tilde{k}_{ii}\tilde{0}_{i,i+1}$ and we conclude that $\tilde{k}_{i+1,i+1}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p$, $q \leqslant n$. Using induction on the index we can then show that $\tilde{k}_{jj}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p$, $q \leqslant n$, for all $i \leqslant j \leqslant n$. Since $\tilde{k}_{nn} \,\omega\, \tilde{0}_{nn} \mathscr{R} \tilde{0}_{n1}$ we have $\tilde{k}_{nn}\tilde{0}_{n1} \mathscr{R} \tilde{k}_{nn}$ and $\tilde{k}_{nn}\tilde{0}_{n1} \,\omega\, \tilde{0}_{n1}$ and so $\tilde{k}_{nn}\tilde{0}_{n1} = \tilde{k}_{n1}$. Since $\tilde{k}_{n1} \,\omega\, \tilde{0}_{n1} \,\omega^l\, \tilde{0}_{11}$ we have $\tilde{0}_{11}\tilde{k}_{n1} \mathscr{L} \tilde{k}_{n1}$ and $\tilde{0}_{11}\tilde{k}_{n1} \,\omega\, \tilde{0}_{11}$ and so $\tilde{0}_{11}\tilde{k}_{n1} = \widetilde{(k+1)}_{11}$. Thus $\widetilde{(k+1)}_{11} = \tilde{0}_{11}\tilde{k}_{n1} = \tilde{0}_{11}\tilde{k}_{nn}\tilde{0}_{n1}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p$, $q \leqslant n$. Using the same argument as before we can now show that $\widetilde{(k+1)}_{jj}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p, q \leqslant n$ for all $1 \leqslant j \leqslant n$; more specifically $\widetilde{(k+1)}_{ii}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p$, $q \leqslant n$.

So far we have shown that if $\tilde{k}_{ii}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p, q \leqslant n$, then so is $\widetilde{(k+1)}_{ii}$. The premise of this statement is certainly true for $k = 0$. Using induction on k we can show that every element of $\omega(\tilde{0}_{ii})$ is a product of elements of $\tilde{0}_{pq}$, $1 \leqslant p, q \leqslant n$, and this holds of course for all $1 \leqslant i \leqslant n$.

Let $\tilde{k}_{ij}$ be any idempotent of $\tilde{B}$. Then $\tilde{k}_{ij} \mathscr{R} \widetilde{(k+r^{ij})}_{ii}$ and $\widetilde{(k+r^{ij})}_{ii} \omega^r \tilde{0}_{ij}$. Therefore $\widetilde{(k+r^{ij})}_{ii}\tilde{0}_{ij} \mathscr{R} \widetilde{(k+r^{ij})}_{ii}$ and $\widetilde{(k+r^{ij})}_{ii}\tilde{0}_{ij}\ \omega\ \tilde{0}_{ij}$ and so $\tilde{k}_{ij} = \widetilde{(k+r^{ij})}_{ii}\tilde{0}_{ij}$ is a product of elements $\tilde{0}_{pq}$, $1 \leqslant p, q \leqslant n$. This proves that $\{\tilde{0}_{pq} \mid 1 \leqslant p, q \leqslant n\}$ is a set of generators for $\tilde{B}$. Since these elements are mapped by the homomorphism θ of Theorem 4.8 onto a minimal set of idempotent generators for the greatest completely simple homomorphic image of $\tilde{B}$, and since θ is one-to-one on $\{\tilde{0}_{pq} \mid 1 \leqslant p, q \leqslant n\}$, the set $\{\tilde{0}_{pq} \mid 1 \leqslant p, q \leqslant n\}$ is a minimal set of idempotent generators for $(\tilde{B}, n, L, R)$.

An important feature of our family $\mathscr{F}$ of building blocks for *BIG* ω-principal semigroups is that every element of $\mathscr{F}$ is a non-completely simple homomorphic image of some $(\tilde{B}, n, L, R)$ whose biordered set (B, n, L, R) we fully understand. A drawback to our family $\mathscr{F}$ is that we have not explicitly constructed the non-completely simple homomorphic images of $(\tilde{B}, n, L, R)$, or even the images of the biordered set (B, n, L, R). In particular we have made no claim that the family $\mathscr{F}$ is a minimal set of building blocks; it is not: in fact, it is easy to see that $(\tilde{B}, n, L, R)/\rho$ and $(\tilde{B}', n', L', R')/\rho'$ may be isomorphic even though $n \neq n'$, $L \neq L'$, $R \neq R'$, $\rho \neq \rho'$.

We conclude the section by pointing out a situation in which a *BIG* ω-principal semigroup requires at most the two building blocks Sp_4 and Sp_4^*.

THEOREM 4.11, *Suppose S is a BIG ω-principal semigroup having Sp_4 as a homomorphic image. Then every $e \in E(S)$ belongs to a subsemigroup of S which is isomorphic to Sp_4 or Sp_4^*.*

Proof. Suppose $\phi: S \twoheadrightarrow Sp_4$, let $e \in E(S)$ and suppose $e_1 \prec e$ (i.e. e covers e_1). By Corollary 4.2 $e_1\phi \prec e\phi$. There is a unique E-chain of length 4 linking $e_1\phi$ and $e\phi$ and we consider two cases depending on the form of this chain.

Case 1. $e\phi = a_1 \mathscr{R} a_2 \mathscr{L} a_3 \mathscr{R} a_4 \mathscr{L} e_1\phi$. Choose $f \in E(S)$ such that $f\phi = a_3$ (such an element f exists by the well-known results of Lallement [4]). Let $g \in S(e, f)$, $h \in S(f, e)$. Then $g\phi = S(e, f)\phi = S(e\phi, f\phi) = S(a_1, a_3) = a_4$ and $h\phi = S(f, e)\phi = S(f\phi, e\phi) = S(a_3, a_1) = a_2$. Since ϕ preserves orientations of sandwich sets we have $e \mathscr{R} g \mathscr{L} f \mathscr{R} h \mathscr{L} e_1$. Hence $T = \langle e, g, f, h\rangle$ is isomorphic to Sp_4 as required.

Case 2. $e\phi = a_1 \mathscr{L} a_2 \mathscr{R} a_3 \mathscr{L} a_4 \mathscr{C} e_1\phi$. In a similar manner we may show that e belongs to a subsemigroup T of S which is isomorphic to Sp_4^*.

5. Coproduct Families

In order to describe all *BIG* ω-principal semigroups which are generated by two copies of the four-spiral semigrop Sp_4 we introduce the concept of a coproduct family.

Definition 5.1. Let $\mathscr{C}$ be a category and let A, B be objects of $\mathscr{C}$. A family of triples $\mathscr{G} = \{(C_i, \alpha_i, \beta_i) \mid i \in I\}$ consisting of objects C_i and morphisms $\alpha_i: A \to C_i$ and $\beta_i: B \to C_i$ is called the *coproduct family of A and B* if for every triple (D, γ, δ) where $\gamma: A \to D$, $\delta: B \to D$ there exists a unique $(C_i, \alpha_i, \beta_i) \in \mathscr{G}$ and a unique $\eta_i: C_i \to D$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[dr, "\alpha_i"'] \arrow[drr, "\gamma"] & & \\
& C_i \arrow[r, "\eta_i"] & D \\
B \arrow[ur, "\beta_i"] \arrow[urr, "\delta"'] & &
\end{tikzcd}$$

Proposition 5.2. *If the coproduct family of A and B exists, then it is unique.*

Proof. Let $\mathscr{G} = \{(C_i, \alpha_i, \beta_i) \mid i \in I\}$ and $\mathscr{H} = \{(E_j, \kappa_j, \lambda_j) \mid j \in J\}$ be coproduct families of A and B. Define a mapping $\psi: \mathscr{G} \to \mathscr{H}$ as follows: given $(C_i, \alpha_i, \beta_i) \in \mathscr{G}$ there exists a unique $(E_j, \kappa_j, \lambda_j) \in \mathscr{H}$ and a unique $\xi_j: E_j \to C_i$ such that

$$\begin{tikzcd}
A \arrow[dr, "\kappa_j"'] \arrow[drr, "\alpha_i"] & & \\
& E_j \arrow[r, "\xi_j"] & C_i \\
B \arrow[ur, "\lambda_j"] \arrow[urr, "\beta_i"'] & &
\end{tikzcd}$$

commutes. Let $\psi((C_i, \alpha_i, \beta_i)) = (E_j, \kappa_j, \lambda_j)$. Similarly we define a mapping $\phi: \mathscr{H} \to \mathscr{G}$, say $\phi((E_j, \kappa_j, \lambda_j)) = (C_k, \alpha_k, \beta_k)$.

$$\begin{tikzcd}
A \arrow[dr, "\alpha_k"'] \arrow[drr, "\kappa_j"] \arrow[drrr, "\alpha_i"] & & & \\
& C_k \arrow[r, "\eta_k"] & E_j \arrow[r, "\xi_j"] & C_i \\
B \arrow[ur, "\beta_k"] \arrow[urr, "\lambda_j"] \arrow[urrr, "\beta_i"'] & & &
\end{tikzcd}$$

By the uniqueness property of $\mathscr{G}$ we have $(C_i, \alpha_i, \beta_i) = (C_k, \alpha_k, \beta_k)$ and $\eta_k \xi_j = 1_{C_i}$. It follows that $\phi \circ \psi$ is the identity mapping on $\mathscr{G}$. Repeating the argument with $\mathscr{G}$ and $\mathscr{H}$ interchanged, we prove that there exists a bijection $\psi \colon \mathscr{G} \to \mathscr{H}$ from $\mathscr{G}$ onto $\mathscr{H}$ such that there is a unique isomorphism η_i from C_i onto $E_j = \psi(C_i)$ which makes

$$\begin{tikzcd}
 & A \arrow[dl, "\alpha_i"'] \arrow[dr, "\kappa_j"] & \\
C_i \arrow[rr, "\eta_i"] & & E_j \\
 & B \arrow[ul, "\beta_i"] \arrow[ur, "\lambda_j"'] &
\end{tikzcd}$$

commute.

Proposition 5.3. *If the coproduct (C, α, β) of A and B exists, then the coproduct family of A and B is $\{(C, \alpha, \beta)\}$.*

Proof. $\{(C, \alpha, \beta)\}$ is the coproduct family by Definition 5.1.

We shall now restrict attention to the category of *BIG* ω-principal semigroups and *BIG* ω-principal semigroup homomorphisms, i.e. homomorphisms with non-completely simple images (see Proposition 4.1). We shall determine the coproduct family of two copies of the four-spiral semigroup Sp_4 in this category.

Let $(C, 4, L, R)$ denote the biordered set (E, n, L, R) with $n = 4$ provided L and R satisfy the following additional restrictions:

$$l^{11} = l^{12} = l^{22} = l^{33} = l^{34} = l^{44} = 0, \qquad l^{21} = l^{43} = 1$$
$$r^{11} = r^{12} = r^{22} = r^{21} = r^{33} = r^{34} = r^{44} = r^{43} = 0.$$

In this case $0_{11} \mathscr{R}\, 0_{12} \mathscr{L}\, 0_{22} \mathscr{R}\, 0_{21} \mathscr{L}\, 1_{11}$ and $0_{33} \mathscr{R}\, 0_{34} \mathscr{L}\, 0_{44} \mathscr{R}\, 0_{43} \mathscr{L}\, 1_{33}$ are both E-chains which generate four-spiral biordered sets. Denote the corresponding four-spirals by $E_4{}^A$ and $E_4{}^B$. Since every element of $(C, 4, L, R)$ is joined by an $\mathscr{L}$ or $\mathscr{R}$ relation to each of these four-spirals, the biordered set $(C, 4, L, R)$ is connected and thus $(C, 4, L, R)$ is the biordered set of a *BIG* ω-principal semigroup.

Example 5.4. The biordered set of $(\tilde{C}, 4, L, R)$, namely $(C, 4, L, R)$, is pictured in Diagram 6 in the case where all of the entries of L and R which are unspecified by the restrictions are zero:

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

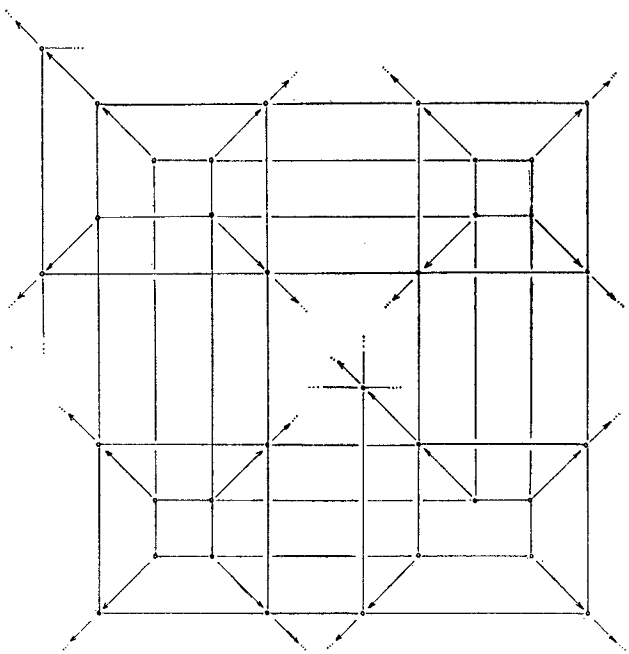

DIAGRAM 6

Let $\tilde{C} = (\tilde{C}, 4, L, R)$ denote the free regular IG-semigroup with respect to $(C, 4, L, R)$. Then $(\tilde{C}, 4, L, R)$ is also a BIG ω-principal semigroup. Denote the four-spiral subsemigroups of $\tilde{C} = (\tilde{C}, 4, L, R)$ generated by $E_4{}^A$ and $E_4{}^B$ by $\tilde{A}(L, R)$ and $\tilde{B}(L, R)$ respectively. If A and B are any two four-spiral semigroups let $\iota_A(L, R)\colon A \to \tilde{A}(L, R)$ and $\iota_B(L, R)\colon B \to \tilde{B}(L, R)$ be the obvious isomorphisms and consider $\iota_A(L, R)$ and $\iota_B(L, R)$ to be homomorphisms from A and B respectively into $(\tilde{C}, 4, L, R)$. Finally we let $\mathscr{G}$ denote the family of all the triples $\{((\tilde{C}, 4, L, R), \iota_A(L, R), \iota_B(L, R))\}$ corresponding to all permissible choices of L and R.

THEOREM 5.5. *$\mathscr{G}$ is the coproduct family of Sp_4 and Sp_4 in the category of BIG ω-principal semigroups.*

Proof. Let $A \simeq Sp_4$, $B \simeq Sp_4$ and let (S, γ, δ) be any triple consisting of a BIG ω-principal semigroup S and morphisms $\gamma\colon A \to S$ and $\delta\colon B \to S$. Consider A as being generated by the elements a, b, c and d subject to the usual relations and B as being generated by the elements a_1, b_1, c_1 and d_1 subject to the usual relations. Let

$$a\gamma = \bar{0}_{11}, \qquad b\gamma = \bar{0}_{12}, \qquad c\gamma = \bar{0}_{22}, \qquad d\gamma = \bar{0}_{21}$$
$$a_1\delta = \bar{0}_{33}, \qquad b_1\delta = \bar{0}_{34}, \qquad c_1\delta = \bar{0}_{44}, \qquad d_1\delta = \bar{0}_{43}.$$

Let $S(\bar{0}_{ii}, \bar{0}_{jj}) = \bar{0}_{ji}$. Note that this is in agreement with our earlier definitions of $\bar{0}_{ji}$ since γ and δ are both bimorphisms when restricted to idempotents of Sp_4. Let $\omega(\bar{0}_{ij}) = \{\bar{k}_{ij} \mid k \in N\}$ where $\bar{0}_{ij} > \bar{1}_{ij} > \cdots$. Let (l^{ij}, r^{ij}) be the orientation of $\bar{0}_{ij}$ with respect to $\bar{0}_{jj}$ and $\bar{0}_{ii}$. Then the matrices $L = (l^{ij})$ and $R = (r^{ij})$ both satisfy the necessary restrictions so that $(C, 4, L, R)$ is a biordered set. By Lemmas 4.5 and 4.7 the mapping θ: $(C, 4, L, R) \to S$ defined by $k_{ij} \to \bar{k}_{ij}$ is a bimorphism, so by Lemma 4.4 it may be extended uniquely to a homomorphism θ: $(\tilde{C}, 4, L, R) \to S$. The fact that $((\tilde{C}, 4, L, R), \iota_A(L, R), \iota_B(L, R))$ is the unique element of $\mathscr{G}$ which makes the required diagram commute follows from Lemma 4.3: θ must preserve sandwich set orientations while distinct $(\tilde{C}, 4, L, R)$'s have distinct sandwich set orientations. Hence $\mathscr{G}$ is the coproduct family of Sp_4 and Sp_4 in the category of *BIG* ω-principal semigroups.

We remark that the theorem implies that if S is any *BIG* ω-principal semigroup which is generated by two copies of Sp_4, then S is a homomorphic image of some semigroup $(\tilde{C}, 4, L, R)$.

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